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A NOTE ON CONSTRUCTION OF VARIANCE BALANCED DESIGNS

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SUMMARY

In this paper variance balanced designs have been constructed in unequal block sizes for situations when suitable *BIB* designs do not exist for a given number of treatments because of the constraints $bk = vr$ and $\lambda (v - 1) = r (k - 1)$.

Keywords: Group Divisible Designs; *BIB* Designs; *A*-efficiency.

Several methods of constructing variance balanced designs with unequal block sizes, unequal replications and even unequal concurrences have been advanced in the literature (John [3], Hedayat and Federer [2], Tyagi [5], Khatri [4], and Agarwal and Kumar [1]). This note gives a method of constructing variance balanced designs for $v + 1$ treatments using Group Divisible (*GD*) designs for v treatments.

Consider a *GD* design D_G with the parameters $v = mn, b, k, r, \lambda_1 = 0, \lambda_2, m, n$. Form m blocks of size n such that within a block all the treatments are first associates. Add a new treatment in each of the m blocks and call the resulting design D_A. Repeat the design D_G α times and the design D_A β times. We then have the following theorem:

THEOREM 1. *The design $D^* = \alpha D_G \cup \beta D_A$ is variance balanced if and only if*

$$\frac{\alpha\lambda_2}{k} = \frac{\beta}{n + 1}, \qquad (1)$$

where α and β are the smallest integers satisfying (1).

Proof. A connected design is variance balanced if and only if the information matrix pertaining to the parameters of interest is of the form

$$C = \theta\,(I - J/v), \tag{2}$$

where I is an identity matrix of order v and J is a $v \times v$ matrix with unitary entries, and θ is a scalar constant.

For design D^*, the constant θ would either be $\theta' = \alpha\lambda_2 v^*/k$ or $\theta'' = \beta v^*/(n+1)$. It, therefore, follows that the design D^* would be variance balanced if and only if (1) holds.

REMARK 1. The parameters of the design D^* are $v^* = v + 1$, $b^* = \alpha_b + \beta_m$, $r_i^* = \alpha_r + \beta (i = 1, 2, \ldots, v)$, $r_{v+1}^* = \beta_m$, $k_j^* = k\,(j = 1, 2, \ldots, b)$, $k_j^* = n + 1\ (j = \alpha_{b+1}, \ldots, \alpha_b + \beta_m)$.

REMARK 2. In the special case when $k = n + 1$, the design D^* would be BIB design with parameters $v^* = v + 1$, $k^* = n + 1$, $\lambda^* = \lambda_2$, $r^* = r + \beta$, $b^* = b + m\beta$.

Consider again designs D_G and D_A. By taking $n + 1$ treatments of each of m blocks of the design D_A, construct m BIB designs each with parameters $\bar{v} = n + 1$, $\bar{k}$, $\bar{\lambda}$, $\bar{b}$, $\bar{r}$ and then define a design D_B as the union of these m BIB designs. Now repeat the original design D_G γ times and the design D_B δ times.

THEOREM 2. *The design* $D^{**} = \gamma D_G \cup \delta D_B$ *is variance balanced if and only if*

$$\frac{\gamma\lambda_2}{k} = \frac{\delta\bar{\lambda}}{\bar{k}}, \tag{3}$$

where γ and δ are the smallest integers satisfying (3).

Proof. It is obvious.

REMARK 3. The design D^{**} would be BIB design if $k = \bar{k}$.

Illustration 1. Consider a GD design D_G with the parameters $v = 12$, $m = 6$, $n = 2$, $b = 8$, $k = 6$, $r = 4$, $\lambda_1 = 0$, $\lambda_2 = 2$ and with block contents (1 2 3 4 5 6), (1 4 6 8 9 11), (2 3 6 7 10 11), (3 4 7 8 11 12), (1 2 9 10 11 12), (1 3 5 8 10 12), (2 4 5 7 9 12), (5 6 7 8 9 10) and design D_A with block contents (1 7 13), (2 8 13), (3 9 13), (4 10 13), (5 11 13), (6 12 13). The equation (1) is satisfied by $\alpha = \beta = 1$. $D^* = D_G \cup D_A$ is variance balanced design and requires 66 units with the same C-matrix as that of the BIB design with parameters $v^* = 13$, $k^* = 3$, $\lambda^* = 1$, $r^* = 6$, $b^* = 26$.

Illustration 2. Consider a *GD* design D_G with parameters $v = 8, m = 4, n = 2, r = 4, b = 8, k = 4, \lambda_1 = 0, \lambda_2 = 2$ with block contents (1 2 3 4), (5 6 7 8), (1 2 7 8), (3 4 5 6), (3 8 1 6), (7 4 5 2), (1 4 6 7), (2 3 5 8), and a design D_B with block contents (1 5), (1 9), (5 9), (2 6), (2 9), (6 9), (3 7), (3 9), (7 9), (4 8), (4 9), (8 9). The equation (3) is satisfied for $\gamma = \delta = 1$. So the design $D^{**} = D_G \cup D_B$ is variance balanced and requires 56 units with the same *C*-matrix as the *BIB* design with parameters $v^* = 9, k^* = 2, r^* = 8, \lambda^* = 1, b^* = 36$.

The overall *A*-efficiency based on all the estimable parametric functions of variance balanced design is given by

$$\text{tr } W^{-1} = \frac{v^2 - v - 1}{vq} = \frac{a_1(b^2 - b + 1)}{b^2} + \frac{a_2}{b} - \frac{a_3(b - 1)}{b^2 q}$$

(see Khatri; [4]).

The overall *A*-efficiency of the usual *BIB* design with parameters $v = 13$, $b = 26, k = 3, r = 6, \lambda = 1$ and the design given by illustration 1 is $\text{tr } W_1^{-1} = 13.0$, and $\text{tr } W_2^{-1} = 6.42$, respectively.

This shows that the design given by illustration 1 is *A*-efficient as compared to the corresponding *BIB* design.

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